

Stability analysis of the cosmological solutions with induced gravity and scalar field on the brane

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Abstract. We study cosmological dynamics and phase space of a scalar field localized on the DGP brane. We consider both the minimally and nonminimally coupled scalar quintessence and phantom fields on the brane. In the nonminimal case, the scalar field couples with induced gravity on the brane. We present a detailed analysis of the critical points, their stability and late-time cosmological viability of the solutions in the phase space of the model.

Key Words: Braneworld Cosmology, Scalar Fields, Dynamical Systems, Cosmological Viability

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1. Introduction

In the revolutionary braneworld viewpoint, our universe is a 3-brane embedded in an extra dimensional bulk. Standard matter and all interactions are confined on the brane; only graviton and possibly non-standard matter are free to probe the full bulk [1]. Based on the braneworld viewpoint, our universe may contain many more dimensions than those we experience with our senses. The most compelling reasons to believe in extra dimensions are that they permit new connections between physical properties of the observed universe and suggest the possibility for explaining some of its more mysterious features. Extra dimensions can have novel implications for the world we see, and they can explain phenomena that seem to be mysterious when viewed from the perspective of a three-dimensional observer. Even if one is doubtful about string theory due to, for instance, its huge number of landscapes, recent researches have provided perhaps the most compelling argument in the favor of extra dimensions: a universe with extra dimensions might contain clues to physics puzzles that have no convincing solutions without them. This reason alone makes extra dimensional theories worthy of investigation. In this streamline, the braneworld models that are inspired by ideas from string theory provide a rich and interesting phenomenology, where higher-dimensional gravity effects in the early and late universe can be explored, and predictions can be made in comparison with high-precision cosmological data. Even for the simplest models of Randall-Sundrum (RS) [2] and Dvali-Gabadadze-Porrati (DGP) [3], braneworld cosmology brings new implications on the inflation and structure formation. Also it brings new ideas for dark energy and opens up exciting prospects for subjecting M-theory ideas to the increasingly stringent tests provided by high-precision astronomical observations. At the same time, braneworld models provide a rich playground for probing the geometry and dynamics of the gravitational field and its interaction with matter. In these respects, the DGP braneworld model is a scenario that gravity is altered at immense distances by the excruciatingly slow leakage of gravity off our 3-brane universe. In this braneworld scenario, the bulk is considered as empty except for a cosmological constant and the matter fields on the brane are considered as responsible for the evolution on the brane [3]. The self-accelerating DGP branch explains late-time speed-up by itself without recourse to dark energy or other mysterious components [4]. Even the normal DGP branch has the potential to realize an effective phantom phase via dynamical screening of the brane cosmological constant [5].

Scalar fields play a crucial role in modern cosmology, both in models of the early universe and late-time acceleration. Scalar fields provide also a simple dynamical model for matter fields in a braneworld and dark energy models. In the early universe, inflaton as a scalar field provides the required basis of the some well-established inflation models. Also at late time, dark energy models based on dynamical scalar fields have been studied extensively in recent years [6]. In braneworld models, the existence of a scalar field on the brane provides a variety of possibilities that brings the corresponding theory to explain some novel properties. In fact, a particular form of the bulk or brane matter is

a scalar field. In the context of braneworld induced gravity, it is natural to consider a non-minimal coupling of the scalar field and induced Ricci curvature on the brane. The resulting theory can be thought of as a generalization of the BransDicke type scalar-tensor gravity in a braneworld context [7]. As has been pointed in [8], the introduction of the non-minimal coupling (NMC) is not just a matter of taste: the NMC is forced upon us in many situations of physical and cosmological interest. For instance, NMC arises at the quantum level when quantum corrections to the scalar field theory are considered. Even if for the classical, unperturbed theory this NMC vanishes, it is necessary for the renormalizability of the scalar field theory in curved space. In most theories used to describe inflationary scenarios, it turns out that a non-vanishing value of the coupling constant is inevitable. In general relativity, and in all other metric theories of gravity in which the scalar field is not part of the gravitational sector, the coupling constant necessarily assumes the value of $\frac{1}{6}$ [8]. Therefore, it is natural to incorporate an explicit NMC between the scalar field and the Ricci scalar in the inflationary paradigm and also in scalar fields models of dark energy. In particular the effect of this NMC in a DGP-inspired braneworld cosmology has been studied by some authors (see [7] and also [9]).

There are several studies focusing on braneworld models with brane/bulk scalar fields. Some of these studies concentrate on the bulk scalar fields minimally or nonminimally coupled to the bulk Ricci scalar [10]. The scalar field minimally or non-minimally coupled to gravity on the brane are studied by some authors [11, 12, 13, 14]. In [14], the authors are studied the self-accelerating solutions in a DGP brane with a scalar field trapped on it within a dynamical system perspective. They have shown that the dynamical screening of the scalar field self-interaction potential occurring within the Minkowski cosmological phase of the DGP model mimics 4D phantom behavior and is an *attractor* solution for a constant self-interaction potential. But, this is not the case necessarily for an exponential potential. For exponential potential, they have shown that gravitational screening is not even a critical point of the corresponding autonomous system of ordinary differential equations. Along with this pioneer work, we consider a scalar field trapped on the DGP brane and we suppose this scalar field is non-minimally coupled to induced gravity on the brane. We study cosmological dynamics on the normal branch of the scenario within a phase space approach with both quintessence and phantom fields on the brane. We provide a phase space analysis of each model through a detailed study of the fixed points, their stability and cosmological viability of the solutions. We also study the classical stability of the solutions in each case in the $w_\varphi - w'_\varphi$ phase-plane. Our study, in comparison with existing literature in this field, provides a complete framework and contains several aspects of the problem not been considered yet. Since the self-accelerating DGP branch has ghost instabilities, our study here is restricted just to the normal DGP branch of the models. While the normal branch of a pure DGP setup has not the potential to explain the late-time cosmic speed-up and crossing of the phantom divide, we show that with a scalar field on the brane there are several new possibilities in the favor of these observationally supported issues.

We note that our motivation to study this model is as follows: as we have indicated above, observations support (at least mildly) that the equation of state parameter of dark energy has crossed the cosmological constant line ($w = -1$) in recent past (at redshift $z \sim 0.25$). It is impossible to realize this feature with a quintessence or phantom field minimally coupled to gravity in standard 4-dimensional theory [6]. Although the original DGP model was proposed to realize accelerated expansion of the universe in a braneworld setup, the self-accelerating branch of the DGP cosmological solutions has ghost instability. It is impossible also to cross the phantom divide line without a scalar field in the self-accelerating DGP branch [11]. The normal DGP solution has no ghost instability, but it cannot explain accelerated expansion and crossing of the phantom divide. It has been shown that localizing a scalar field on the normal DGP setup realizes these features [11]. It is possible also to incorporate extra degrees of freedom on the braneworld setup to have more successful models (see for instance [7]). These extra degrees provide new facilities and richer cosmological history on the brane, a part of which is related to the wider parameter space. On the other hand, considering just a cosmological constant on the brane, although explains accelerated expansion through dynamical screening of the brane cosmological constant, it has not the potential to explain crossing of the phantom divide [5]. In this paper we have shown that a scalar field, minimally or nonminimally coupled to induced gravity on the brane has the potential to fill these gaps. We stress that all of the accelerated phases obtained in this study belong to the normal DGP branch of the model. Our study, based on the phase space analysis, provides the most complete treatment of the issue in the field. We have provided a complete analysis of the problem focusing on all possible details.

2. Cosmological dynamics of a minimally coupled scalar field on the DGP brane

DGP braneworld scenario has attracted a lot of attention through these years. Although this scenario has very interesting phenomenological aspects, it suffers from shortcomings such as ghost instabilities in its self-accelerating branch of the solutions. Fortunately the normal, non-self-accelerating branch of the DGP cosmological solutions has no ghost instabilities. For this reason we consider a scalar field on the DGP brane and we analyze cosmological dynamics of the *normal DGP* solutions in this setup. To begin and in order to explain the frame of our analysis, we start with the case of a minimally coupled quintessence field on the normal DGP branch. We note that this problem has been considered previously in [11, 12, 14]. Especially, in [14] the authors have presented a detailed study of the problem with just a minimally coupled scalar field on the brane. We firstly study quintessence field with more details and more enlightening analysis than [14] both in calculations and corresponding analysis of the phase space points. Then we extend our study to the minimally coupled phantom fields, nonminimally coupled quintessence fields and finally non-minimally coupled phantom fields on the brane. In each step, we provide a detailed analysis of the model in phase space and

within a dynamical system approach. We study the late-time cosmological viability of the solutions and their stability in each step. We also investigate the classical stability of the solutions in $w_\varphi - w'_\varphi$ phase-plane.

2.1. Minimally coupled quintessence field on the brane

A minimally coupled quintessence field on the normal DGP within a phase space approach first has been considered in [11]. There, the authors have considered the cosmological evolution of a QDGP model in a phantom-like prescription. They have defined an effective cosmological constant that increases by time evolution of the Hubble parameter and therefore realizes a phantom-like behavior through dynamical screening of the brane cosmological constant. Here we adopt another strategy: we focus on the phase space coordinates instead of the brane cosmological constant similar to strategy adopted in [14]. Although in this step the results are the same, but our analysis in this subsection introduces the notation, conventions and general framework of our procedure. In addition, some novel ingredients such as $w_\varphi - w'_\varphi$ phase-plane stability analysis, more detailed calculations and more enlightening plots are presented.

The action of an induced gravity braneworld model can be written as follows (we use the sign convention of [6])

$$\mathcal{S} = \frac{-M_5^3}{2} \int d^5X \sqrt{-g} R_5 - \frac{\mu^2}{2} \int d^4x \sqrt{-h} R_4 + \int d^4x \sqrt{-h} \mathcal{L}_m + \mathcal{S}_{GH}, \quad (1)$$

where g_{ab} is the metric of the bulk manifold and $h_{\mu\nu}$ is the induced metric on the brane. R_5 and R_4 denote the 5 and 4 dimensional Ricci scalars respectively, and \mathcal{L}_m is the matter Lagrangian confined on the brane. \mathcal{S}_{GH} is the Gibbons-Hawking boundary action which is required in order to apply the boundary conditions properly. In this induced gravity braneworld setup, the ratio of the two scales, the 4-dimensional Planck mass μ and its 5-dimensional counterpart M_5 , defines the DGP crossover scale as

$$r_c = \frac{\mu^2}{2M_5^3},$$

which determines the behavior of gravity in different distance scales on the brane. Adopting a FRW line element, the cosmology of the model is based on the following Friedmann equation [4, 11]

$$H^2 + \frac{K}{a^2} = \left(\sqrt{\frac{\rho}{3\mu^2} + \frac{1}{4r_c^2}} + \frac{1}{2r_c} \right)^2, \quad (2)$$

where ρ is the energy density of the total cosmic fluid on the brane and consists of the energy densities of the scalar field and ordinary matter on the brane. We consider the flat geometry in which ($K = 0$), so the above equation reduces to

$$H^2 + \frac{H}{r_c} = \frac{\rho}{3\mu^2}, \quad (3)$$

for the *normal* DGP branch of the scenario. When the Hubble length H^{-1} is much smaller than r_c , which stands for the early time, the term $\frac{H}{r_c}$ can be ignored relative

to the first term on the left hand side of (3). This term becomes important on scales comparable to the crossover scale when H^{-1} is larger than r_c , which corresponds to the late-time of the universe evolution. We emphasize that in which follows we focus just on the normal, non-self-accelerating DGP branch of the solutions. This is because the normal branch has no ghost instabilities. Nevertheless, the pure normal branch cannot explain the late-time acceleration without additional components on the brane. However, in the presence of a scalar field on the brane, it is possible to realize the late-time cosmic speed-up even in the normal DGP branch [11, 14]. This is the reason why we considered an extension of the DGP setup with a scalar field on the brane.

By differentiation of (3) with respect to the cosmic time, we obtain an equation for evolution of the Hubble parameter

$$\dot{H} = \frac{-(\rho + p)}{2\mu^2} \left(1 + \frac{1}{2r_c H}\right)^{-1}, \quad (4)$$

where we have used the continuity equation on the brane as $\dot{\rho} + 3H(\rho + p) = 0$. We note that the negativity of \dot{H} ensures the phantom-like behavior on the brane [11]. The dynamics of the scalar field localized on the brane is given by the following Klein-Gordon equation

$$\ddot{\varphi} + 3H\dot{\varphi} = -\frac{dV}{d\varphi}. \quad (5)$$

The energy density and pressure of the total matter localized on the brane are given by

$$\rho = \rho_\varphi + \rho_m = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) + \rho_m, \quad (6)$$

$$p = p_\varphi + p_m = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) + w_m \rho_m \quad (7)$$

respectively. To translate our equations of the cosmological dynamics in the language of the autonomous dynamical system, we introduce the following dimensionless quantities

$$\begin{aligned} x_1 &= \frac{\dot{\varphi}}{\sqrt{6\mu H}}, & x_2 &= \frac{\sqrt{V}}{\sqrt{3\mu H}}, & x_3 &= \frac{\sqrt{\rho_m}}{\sqrt{3\mu H}}, \\ x_4 &= \frac{1}{\sqrt{2r_c H}}, & \lambda &= -\frac{V'\mu}{V}, & \Gamma &= \frac{VV''}{V'^2}, \end{aligned} \quad (8)$$

where a prime marks differentiation with respect to the scalar field, $\prime \equiv \frac{d}{d\varphi}$. Through this paper we consider the exponential potential of the scalar field as $V(\varphi) = V_0 e^{-\lambda\varphi/\mu}$. This potential corresponds to a constant λ and gives $\Gamma = 1$. The case of a constant potential (as has been studied separately in [14]), is a special case of this potential. We note that the above dimensionless quantities have explicit physical origin: x_1^2 is related to the kinetic energy of the field, x_2^2 is related to the potential energy of the scalar field, x_3^2 is related to the ordinary matter density on the brane, x_4^2 reflects the DGP character of this setup, and λ and Γ are actually the slow-roll parameters of the model. Our main equations are the Friedmann equation that appears as a constraint, the Klein-Gordon equation and the continuity equation on the brane. These equations with the above dimensionless quantities provide the basis of our dynamical system analysis.

By using the above dimensionless quantities, we rewrite (5), (6) and (7) in the following form

$$\ddot{\phi} = -3\mu H^2(\sqrt{6}x_1 - \lambda x_2^2), \quad (9)$$

$$\rho = 3\mu^2 H^2(x_1^2 + x_2^2 + x_3^2), \quad (10)$$

$$p = 3\mu^2 H^2(x_1^2 - x_2^2 + w_m x_3^2). \quad (11)$$

The effective equation of state parameter in this case is given by

$$w_{eff} = \frac{x_1^2 - x_2^2 + w_m x_3^2}{x_1^2 + x_2^2 + x_3^2}. \quad (12)$$

We can obtain the exact cosmological solutions at the critical points by using the following relation

$$\dot{H} = -\frac{3}{2} \frac{[2x_1^2 + (1 + w_m)x_3^2]}{1 + x_4^2} H^2. \quad (13)$$

In each fixed point, we can rewrite this relation in the following form

$$\dot{H} = -\frac{1}{\alpha} H^2, \quad (14)$$

where by definition

$$\alpha = \frac{2(1 + x_4^2)}{3(2x_1^2 + (1 + w_m)x_3^2)}, \quad \alpha \neq 0. \quad (15)$$

An integration of (14) with respect to the cosmic time gives

$$a(t) = a_0(t - t_0)^\alpha \quad (16)$$

which corresponds to an accelerating phase if $\alpha > 1$. Note that equation (16) is valid only in a small neighborhood around the critical point where α can be considered to be nearly constant. Now the Friedmann constraint equation in terms of the dimensionless quantities becomes

$$x_1^2 + x_2^2 + x_3^2 - 2x_4^2 = 1. \quad (17)$$

So, the allowable region of the phase space is actually outside of a unit sphere defined as $x_1^2 + x_2^2 + x_3^2 = 1$. Now the autonomous dynamical equations are given as follows

$$\frac{dx_1}{dN} = -3x_1 + \frac{\sqrt{6}}{2} \lambda x_2^2 + 3x_1 \left(\frac{2x_1^2 + (1 + w_m)x_3^2}{1 + x_1^2 + x_2^2 + x_3^2} \right), \quad (18)$$

$$\frac{dx_2}{dN} = -\frac{\sqrt{6}}{2} \lambda x_1 x_2 + 3x_2 \left(\frac{2x_1^2 + (1 + w_m)x_3^2}{1 + x_1^2 + x_2^2 + x_3^2} \right), \quad (19)$$

$$\frac{dx_3}{dN} = -\frac{3}{2} (1 + w_m) x_3 + 3x_3 \left(\frac{2x_1^2 + (1 + w_m)x_3^2}{1 + x_1^2 + x_2^2 + x_3^2} \right), \quad (20)$$

where by definition, $N = \ln a(t)$. The eigenvalues of the Jacobian matrix are as follows

Table 1. Location and dynamical character of the fixed points.

| name | x_{1c} | x_{2c} | x_{3c} | Existence | stability | Ω_φ | γ_φ | w_{eff} | $a(t)$ |
|-----------|---|---|--|---------------------------|--|-----------------------------|-----------------------|---------------------------|------------------------------|
| (1a),(1b) | 0 | 0 | ± 1 | $\forall \lambda, \gamma$ | saddle point | 0 | undefined | $\gamma - 1$ | $a_0(t - t_0)^{2/3\gamma}$ |
| (2a) | 1 | 0 | 0 | $\forall \lambda, \gamma$ | saddle point for $\lambda > \sqrt{6}$ unstable node for $\lambda < \sqrt{6}$ | 1 | 2 | 1 | $a_0(t - t_0)^{1/3}$ |
| (2b) | -1 | 0 | 0 | $\forall \lambda, \gamma$ | unstable node for $\lambda > -\sqrt{6}$ saddle point for $\lambda < -\sqrt{6}$ | 1 | 2 | 1 | $a_0(t - t_0)^{1/3}$ |
| (3a),(3b) | $\frac{\lambda}{\sqrt{6}}$ | $\pm(1 - \frac{\lambda^2}{6})^{1/2}$ | 0 | $\lambda^2 \leq 6$ | saddle point | 1 | $\frac{\lambda^2}{3}$ | $\frac{\lambda^2}{3} - 1$ | $a_0(t - t_0)^{2/\lambda^2}$ |
| (4a),(4b) | $\sqrt{\frac{3}{2}} \frac{\gamma}{\lambda}$ | $\pm(\frac{3\gamma(2-\gamma)}{2\lambda^2})^{1/2}$ | $\pm(1 - \frac{3\gamma}{\lambda^2})^{1/2}$ | $\lambda^2 > 3\gamma$ | stable node for $3\gamma < \lambda^2 < \frac{24\gamma^2}{9\gamma-2}$ stable spiral for $\lambda^2 > \frac{24\gamma^2}{9\gamma-2}$ | $\frac{3\gamma}{\lambda^2}$ | γ | $\gamma - 1$ | $a_0(t - t_0)^{2/3\gamma}$ |

- points 1a , 1b :

$$\alpha_1 = \frac{3}{2}\gamma, \quad \alpha_2 = \frac{3}{2}\gamma, \quad \alpha_3 = -\frac{3}{2}(2 - \gamma).$$

- point 2a :

$$\alpha_1 = 3, \quad \alpha_2 = -\frac{\sqrt{6}}{2}\lambda + 3, \quad \alpha_3 = \frac{3}{2}(2 - \gamma).$$

- point 2b :

$$\alpha_1 = 3, \quad \alpha_2 = \frac{\sqrt{6}}{2}\lambda + 3, \quad \alpha_3 = \frac{3}{2}(2 - \gamma).$$

- points 3a , 3b :

$$\alpha_1 = \frac{\lambda^2}{2}, \quad \alpha_2 = \frac{\lambda^2}{2} - 3, \quad \alpha_3 = \frac{1}{2}(\lambda^2 - 3\gamma).$$

- points 4a , 4b , 4c , 4d :

$$\alpha_1 = \frac{3}{2}\gamma, \quad \alpha_{2,3} = \frac{-3}{4}(2 - \gamma) \left(1 \pm \sqrt{1 - \frac{8\gamma(\lambda^2 - 3\gamma)}{\lambda^2(2 - \gamma)}} \right),$$

where $\gamma \equiv 1 + w_m$ is the barotropic index which depends on the type of ordinary matter on the brane. In what follows, we consider $0 < \gamma < 2$.

Table 1 summarizes the complete information about existence, stability and cosmological characteristics of these phase points. In this table, $\Omega_\varphi = \frac{\rho_\varphi}{\rho_c}$ where $\rho_c = 3\mu^2 H^2$ and $\gamma_\varphi = 1 + w_\varphi$. We note that for point 1 with $\gamma = 0$, there is a stable center. All other mentioned points have center too, depending on the values of γ and λ . But these centers are not necessarily stable. For instance, for point $\{(3a),(3b)\}$, the center is unstable.

For points $\{(1a),(1b)\}$, there is no contribution of the scalar field and the universe is dominated by other matter fields. According to the eigenvalues and since $0 < \gamma < 2$, these points behave like saddle points in the phase space.

Points $\{(2a),(2b)\}$ are solutions dominated by the kinetic energy of the scalar field. The contribution of scalar field potential and the energy densities of other matter fields are irrelevant for these phases. For these points, we obtain $\gamma_\varphi = 2$, which is referred to as a *stiff matter* equation of state. These points have no late time acceleration and the

Table 2. Location and critical point for $\gamma_\varphi = 0$ (that is, $w_\varphi = -1$).

| name | x_{2c} | x_{3c} | x_{4c} | stability | w_{eff} | $a(t)$ |
|-----------|-------------------------|----------|----------|------------------|--------------|---|
| (1a),(1b) | 0 | ± 1 | 0 | unstable nodes | $\gamma - 1$ | $(t - t_0)^{2/3\gamma}$ |
| (2a),(2b) | $\pm(1 + 2x_4^2)^{1/2}$ | 0 | x_4 | stable attractor | -1 | $e^{\Lambda(t-t_0)}$ ($\Lambda = constant$) |

stability of them depends on the values of λ , so that for $\lambda < \sqrt{6}$ (for point (2a)) and $\lambda > -\sqrt{6}$ (for point (2b)), we obtain unstable nodes. Otherwise they are saddle points.

For critical points $\{(3a), (3b)\}$, the energy density of universe is dominated by the scalar field's kinetic and potential energies. For $\lambda^2 < 2$ we have accelerated phase of expansion, but this phase is not stable. In these cases, for $\lambda = 0$, the universe is dominated by a cosmological constant.

The last line of table 1 contains four critical points $\{(4a), (4b), (4c), (4d)\}$, so that depending on the values of γ and λ , we have stable spirals or stable nodes. Here γ_φ is equal to the barotropic index of matter, γ . This is a reflection of the fact that the exponential potential used in this framework can give rise to an accelerated expansion and possesses cosmological scaling solutions in which the field energy density ρ_φ is proportional to the fluid energy density, ρ_m .

The ingredients of table 1 can be explained with more geometrical details through the phase space trajectories. Figure 1 shows the two dimensional $(x_1 - x_2)$ phase plane for $\lambda = +1$. In this figure, points A and B both are unstable nodes.

We note that the above arguments were based on the assumption that $x_4 = 0$ for all critical points. Since x_4 is related directly to the braneworld nature of the solutions, our analysis up to this point was effectively 4-dimensional. Now we consider the case that $x_4 \neq 0$. Table 2 gives the results of the corresponding phase space analysis. In this case, there is a cosmological constant dominated accelerating phase which is an stable attractor corresponding to critical lines (2a), (2b) of table 2 (curves C_1 and C_2 of figure 2).

As we have stated previously, the pure normal DGP solution has not the potential to explain the late-time acceleration. In our case with a quintessence field on the brane, as we have shown, it is possible to realize the late-time acceleration in this setup. We note also that as table 1 shows, for fixed points $\{(1a), (1b)\}$, $\{(3a), (3b)\}$ and also $\{(4a), (4b)\}$ it is possible in principle to realize a late-time acceleration which depends on parameters γ and λ and the stability of which needs to be investigated in each case. For fixed points $\{(1a), (1b)\}$ (first row of table 1), the accelerating phase (with $q < 0$ where $q \equiv -\frac{\ddot{a}}{a^2}$) is possible if $\gamma < 2/3$. But this accelerating phase is a saddle point and obviously is not the late-time cosmic accelerating phase. For fixed point $\{(3a), (3b)\}$, the accelerating phase is possible if $\lambda^2 < 2$. This gives also a saddle point which is not corresponding to the late-time stable, accelerating phase of cosmic expansion. For fixed point $\{(4a), (4b), (4c), (4d)\}$, there is an accelerating, stable phase if $\gamma < 2/3$. However, these stable points are either a node or a spiral. Albeit these are not corresponding to a late-time accelerating,

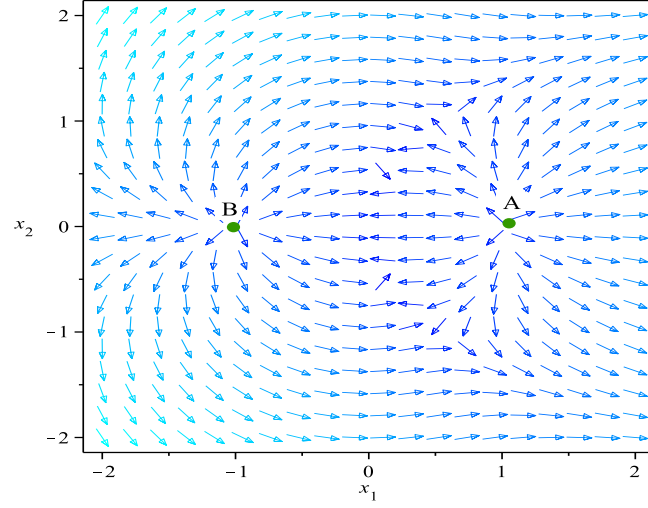


Figure 1. The phase plane for $\lambda = +1$. Points A and B are unstable nodes.

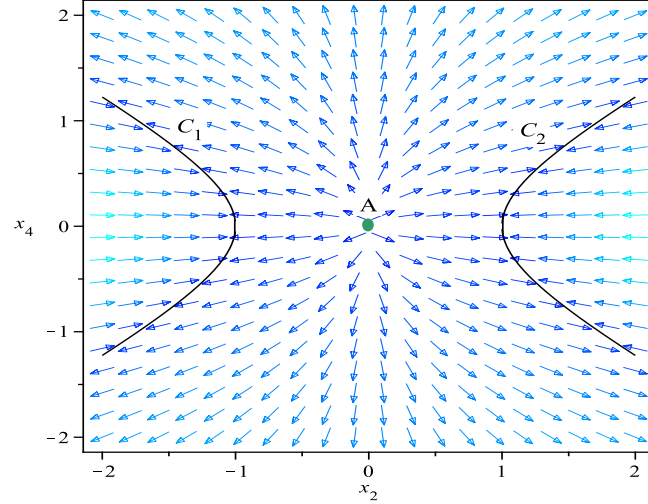


Figure 2. The phase plane for $w_m = 0$. Point A is an unstable node (which reflects the first line of table 2). There are two curves (C_1 and C_2), which are corresponding to critical lines ($(2a)$, $(2b)$) of table 2.

stable, de Sitter attractor. So, none of the accelerating phases corresponding to table 1 are the de Sitter phase. Nevertheless, as we have stated previously, point (2) (actually a critical line) of table 2 gives a de Sitter attractor. Therefore, with a quintessence field in the normal DGP setup, it is possible to realize the late-time acceleration in contrast to pure DGP case.

Now we analyze the accelerating phase of the model through the evolution of the deceleration parameter. The equation for the deceleration parameter is $q = -\frac{\ddot{a}a}{\dot{a}^2} =$

$\frac{1}{2}(1 + 3w_{eff})$. In our case, it can be written by using the phenomenological parameters Ω_m and Ω_φ as

$$q = \frac{1}{2} \frac{(1 + 3w_m)\Omega_m(1+z)^{3(1+w_m)} + (1 + 3w_\varphi)\Omega_\varphi(1+z)^{3(1+w_\varphi)}}{\Omega_m(1+z)^{3(1+w_m)} + \Omega_\varphi(1+z)^{3(1+w_\varphi)}}. \quad (21)$$

Figure 3 shows the behavior of q versus the redshift. There is a transition to the accelerating phase at $z = 0.68$.

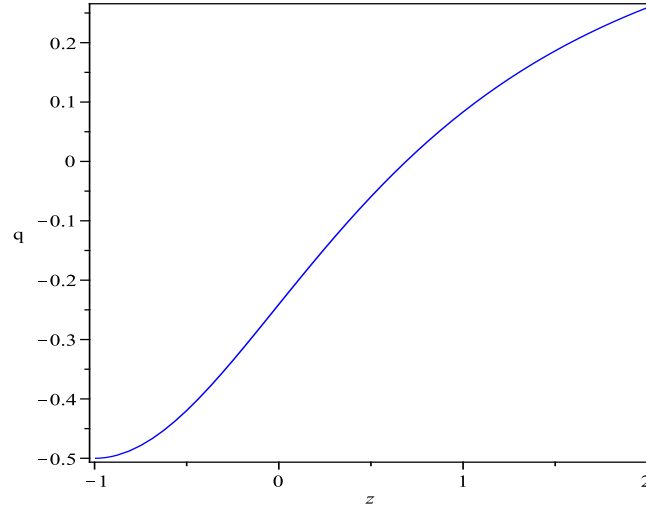


Figure 3. The deceleration parameter versus the redshift for $\Omega_m = 0.28$, $\Omega_\varphi = 0.8$, $w_m = 0$ and $w_\varphi = -2/3$. Transition to the accelerating phase occurs at $z = 0.68$.

We define the dimensionless density parameters as follows

$$\Omega_m = \frac{\rho_0}{3H_0^2}, \quad \Omega_\varphi = \frac{\rho_{\varphi 0}}{3H_0^2}, \quad \Omega_{r_c} = \frac{1}{4r_c^2 H_0^2}.$$

Then the Friedmann equation can be rewritten as

$$H(z) = H_0 \left(\sqrt{\Omega_{r_c} + \Omega_m(1+z)^{3(1+w_m)} + \Omega_\varphi(1+z)^{3(1+w_\varphi)}} - \sqrt{\Omega_{r_c}} \right) \quad (22)$$

Figure 4 shows the evolution of the Hubble parameter versus the redshift and equation of state parameter of a minimally coupled quintessence scalar field with $\Omega_m = 0.28$, $\Omega_\varphi = 0.8$. We can understand from this figure that the Hubble parameter of the model decreases as the redshift decreases. This is a trace of essentially possible realization of an *effective* phantom-like behavior on the brane (see [11] for more details). Figure 5 is a 2-dimensional plot of H versus the redshift for a quintessence field with $w_\varphi = -0.8$.

Another important issue is the possibility of crossing of the cosmological constant line (the so-called phantom-divide line) by the equation of state parameter. The effective equation of state parameter can be written as follows

$$w_{eff} = \frac{w_m\Omega_m(1+z)^{3(1+w_m)} + w_\varphi\Omega_\varphi(1+z)^{3(1+w_\varphi)}}{\Omega_m(1+z)^{3(1+w_m)} + \Omega_\varphi(1+z)^{3(1+w_\varphi)}}. \quad (23)$$

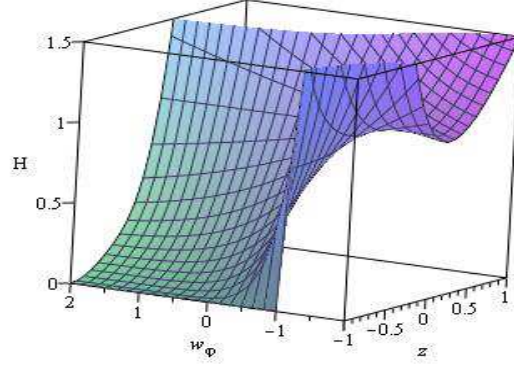


Figure 4. The 3-dimensional plot of the Hubble parameter versus the redshift and equation of state parameter of the scalar field.

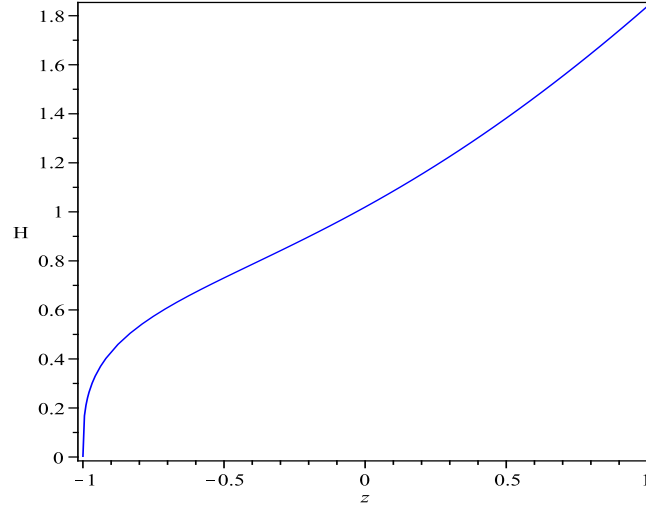


Figure 5. This figure shows the plot of the Hubble parameter versus the redshift for a quintessence field. It is plotted for $\Omega_m = 0.28$, $\Omega_\varphi = 0.8$, $w_m = 0$ and $w_\varphi = -0.8$.

As figure 6 shows, it is impossible to cross the cosmological constant equation of state parameter $w_\varphi = -1$ by a minimally coupled quintessence field in the normal DGP setup.

Now we focus on the classical stability of the solutions in $w_\varphi - w'_\varphi$ phase-plane of the present model (see [15] for a similar analysis for other interesting cases). Here a prime marks the derivative of w_φ with respect to the logarithm of the scale factor, $N = \ln a(t)$, so that

$$w'_\varphi \equiv \frac{dw_\varphi}{dN} = \frac{dw_\varphi}{d\rho_\varphi} \frac{d\rho_\varphi}{dN}. \quad (24)$$

We define the function c_a so that $c_a^2 \equiv \frac{\dot{p}_\varphi}{\dot{\rho}_\varphi}$ or equivalently $c_a^2 \equiv \frac{dp_\varphi}{d\rho_\varphi}$. Generally the

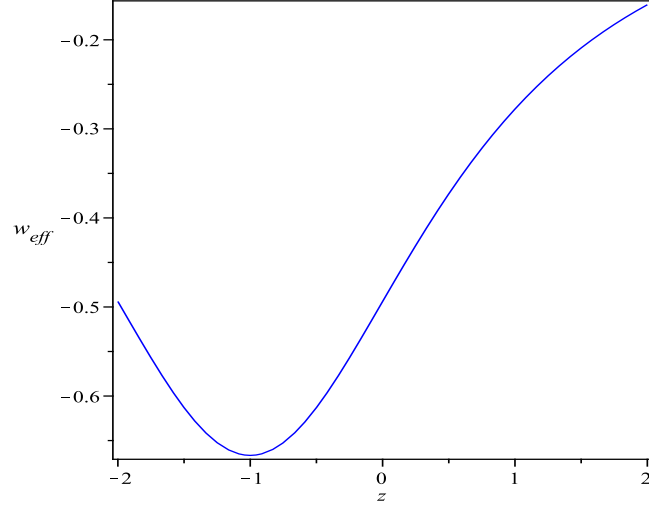


Figure 6. The effective equation of state parameter versus the redshift for $\Omega_m = 0.28$, $\Omega_\varphi = 0.8$, $w_m = 0$ and $w_\varphi = -2/3$. There is no crossing of the cosmological constant line.

sound speed expresses the phase velocity of the inhomogeneous perturbations of the scalar field. If we suppose the scalar field's energy-momentum to have a perfect fluid form, this function would be the adiabatic sound speed of this fluid. To avoid the future big rip singularity, we set $c_a^2 > 0$. Since

$$\begin{aligned} \frac{dw_\varphi}{d\rho_\varphi} &= \frac{1}{\rho_\varphi} \frac{dp_\varphi}{d\rho_\varphi} - \frac{p_\varphi}{\rho_\varphi^2} \\ &= \frac{1}{\rho_\varphi} \left(\frac{dp_\varphi}{d\rho_\varphi} - w_\varphi \right), \end{aligned} \quad (25)$$

and

$$\frac{d\rho_\varphi}{dN} = \frac{\dot{\rho}_\varphi}{H} = -3(1 + w_\varphi)\rho_\varphi, \quad (26)$$

we obtain

$$w'_\varphi = -3(1 + w_\varphi)(c_a^2 - w_\varphi). \quad (27)$$

Therefore, we obtain

$$w'_\varphi = -3(1 - w_\varphi^2) + \lambda\sqrt{3(1 + w_\varphi)\Omega_\varphi(1 - w_\varphi)}. \quad (28)$$

Now the $w_\varphi - w'_\varphi$ phase plane is divided into the following two regions

$$\begin{cases} w_\varphi > -1, & w'_\varphi < 3w_\varphi(1 + w_\varphi) & c_a^2 > 0 & (\text{region I}) \\ w_\varphi > -1, & w'_\varphi > 3w_\varphi(1 + w_\varphi) & c_a^2 < 0 & (\text{region II}) \end{cases} \quad (29)$$

that are shown in figure 7. The region I is the classical stability region of the theory. We note that there is no phantom phase in this case and therefore we have not encounter with four distinct regions of $w_\varphi - w'_\varphi$ plane as usually are discussed in literature.

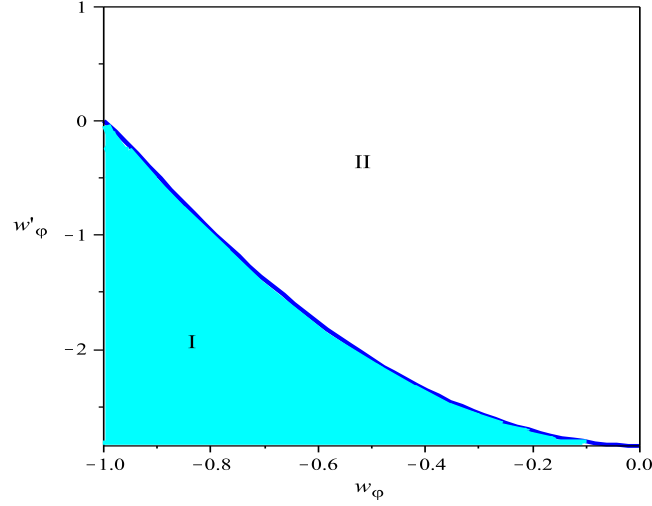


Figure 7. Bounds on w'_φ as a function of w_φ in $w_\varphi - w'_\varphi$ phase-plane for $\Omega_\varphi = 0.8$ and $\lambda = 0.1$

2.2. Minimally coupled phantom field on the normal DGP setup

Astrophysical data indicate that w , the equation of state parameter of the cosmic fluid, lies in a very narrow strip close to -1 . The case $w = -1$ corresponds to the cosmological constant. For w less than -1 , the *phantom* dark energy is observed, and for w more than -1 (but less than $-\frac{1}{3}$) the dark energy is described by a quintessence field which has been studied in the previous subsection. Moreover, the analysis of the properties of dark energy from recent observational data mildly favor models of dark energy with w crossing the -1 line in the near past. So, the phantom phase equation of state with $w < -1$ is mildly allowed by observations [16, 17, 18]. In this case, the universe currently lives in its phantom phase which ends eventually at a future singularity (the Big Rip singularity [19]). There are also a lot of evidence all around of a dynamical equation of state, which has crossed the so called the phantom divide line $w = -1$ recently, at the value of redshift parameter $z \approx 0.25$ [16, 17, 18]. For these reasons, now we pay our attention to a phantom field localized on the DGP brane and we study cosmological dynamics of the normal DGP branch in this case within a phase space analysis.

The energy density and pressure of a phantom field are defined as

$$\rho_\varphi = -\frac{1}{2}\dot{\varphi}^2 + V(\varphi), \quad (30)$$

$$p_\varphi = -\frac{1}{2}\dot{\varphi}^2 - V(\varphi), \quad (31)$$

respectively. The Klein-Gordon equation governing on the dynamics of the phantom field is given by

$$\ddot{\varphi} + 3H\dot{\varphi} = \frac{dV}{d\varphi}. \quad (32)$$

Table 3. Location and dynamical character of the fixed points.

| name | x_{1c} | x_{2c} | x_{3c} | Existence | stability | Ω_φ | γ_φ | w_{eff} | $a(t)$ |
|-----------|---|---|--|------------------------------------|--------------|------------------------------|------------------------|----------------------------|------------------------------------|
| (1a),(1b) | 0 | 0 | ± 1 | $\forall \lambda, \gamma$ | saddle point | 0 | undefined | $\gamma - 1$ | $a_0(t - t_0)^{2/3\gamma}$ |
| (2a),(2b) | $-\frac{\lambda}{\sqrt{6}}$ | $\pm(1 + \frac{\lambda^2}{6})^{1/2}$ | 0 | $\forall \lambda, \gamma$ | stable node | 1 | $\frac{-\lambda^2}{3}$ | $\frac{-\lambda^2}{3} - 1$ | $a_0(t - t_0)^{-2/\lambda^2}$ |
| (3a),(3b) | $\sqrt{\frac{3}{2}} \frac{\gamma}{\lambda}$ | $\pm(\frac{3\gamma(\gamma-2)}{2\lambda^2})^{1/2}$ | $\pm(1 + \frac{3\gamma}{\lambda^2})^{1/2}$ | $\gamma < 0, \lambda^2 > -3\gamma$ | saddle point | $-\frac{3\gamma}{\lambda^2}$ | γ | $\gamma - 1$ | $a_0(t - t_0)^{\frac{2}{3\gamma}}$ |

With phase space coordinates as defined in (8), The Friedmann equation of the model in phase space now is written as

$$-x_1^2 + x_2^2 + x_3^2 - 2x_4^2 = 1, \quad (33)$$

and the autonomous dynamical equations are

$$\frac{dx_1}{dN} = -3x_1 - \frac{\sqrt{6}}{2}\lambda x_2^2 + 3x_1 \left(\frac{-2x_1^2 + (1+w)x_3^2}{1 - x_1^2 + x_2^2 + x_3^2} \right), \quad (34)$$

$$\frac{dx_2}{dN} = -\frac{\sqrt{6}}{2}\lambda x_1 x_2 + 3x_2 \left(\frac{-2x_1^2 + (1+w)x_3^2}{1 - x_1^2 + x_2^2 + x_3^2} \right), \quad (35)$$

$$\frac{dx_3}{dN} = -\frac{3}{2}x_3 \left(\frac{-2x_1^2 + (1+w)x_3^2}{1 - x_1^2 + x_2^2 + x_3^2} \right). \quad (36)$$

In this case there are eight critical points that are shown in table 3. The eigenvalues of these points are

- points (1a) , (1b):

$$\alpha_1 = \frac{3}{2}(\gamma - 2), \quad \alpha_2 = \frac{3}{2}\gamma, \quad \alpha_3 = \frac{3}{2}\gamma$$

- points (2a) , (2b):

$$\alpha_1 = -\frac{1}{2}(\lambda^2 + 6), \quad \alpha_2 = -\frac{1}{2}(\lambda^2 + 3\gamma), \quad \alpha_3 = -\frac{\lambda^2}{2}$$

- points (3a) , (3b) , (3c) , (3d):

$$\alpha_1 = \frac{3}{2}\gamma, \quad \alpha_{2,3} = -\frac{3}{4}(2 - \gamma) \left(1 \pm \sqrt{1 - \frac{8\gamma(\lambda^2 + 3\gamma)}{\lambda^2(2 - \gamma)}} \right).$$

In table 3 we summarized also the results of the phase space analysis of the model in addition to cosmological characters of each critical point. For critical points $\{(1a), (1b)\}$, there is no contribution of the phantom scalar field and the universe is dominated by matter fields other than the phantom scalar field. These points behave like saddle points in the phase space and for $\gamma < 2/3$, these points give an accelerating phase. It is possible to have scaling solutions in this case too. For points $\{(2a), (2b)\}$, there is no contribution of ordinary matter fields and the energy density of the universe is dominated by the phantom scalar field's kinetic and potential energies. In these cases,

there is no possibility of accelerated expansion on the brane. The corresponding points in phase plane are stable nodes. The last line of table 3 consists of four critical points $\{(3a), (3b), (3c), (3d)\}$. These are just saddle points in phase plane. There are scaling solutions in these critical points. For these fixed points accelerated expansion is possible if $\gamma < 2/3$. However, this accelerated expansion phase is not a de Sitter stable phase.

We note that the analysis presented in the previous paragraph was based on the condition $x_4 = 0$ which gives essentially an effective 4-dimensional picture of the model. Now we consider the case that $x_4 \neq 0$. The phase space analysis of the model with minimally coupled phantom field gives *the same* results as are presented in table 2 for a minimally coupled quintessence field. Similar to minimal quintessence scalar field case, in this case there is a cosmological constant dominated accelerating phase which is an stable attractor corresponding to critical lines (2a), (2b) of table 2 (curves C_1 and C_2 of figure 2). So, with a phantom field minimally coupled to induced gravity in the normal DGP setup it is possible to have an attractor, de Sitter solution realizing the late-time accelerated expansion (see table 2). Figure 8 shows a plot of the $x_1 - x_2$ phase plane of the model. Point A is corresponding to points (1a) and (1b) of table 3. Points B and C are corresponding to points (2a) and (2b) of table 3 which are stable nodes. The $x_3 - x_2$ (with $x_4 \neq 0$) phase plane of the model is shown in figure 9. We note that this figure is actually the same as figure 2 but now plotted in $x_3 - x_2$ plane rather than $x_2 - x_4$ plane. Here we encounter a line of stability points as is shown in figure 9. Points A and B are corresponding to critical points (1a) and (1b) of table 2.

The form of the deceleration parameter and effective equation of state parameter for phantom field are the same as quintessence field presented as equations (21) and (23). Figure 10 shows the behavior of the deceleration parameter $q(z)$. The universe enters the accelerated phase at $z \simeq 0.68$. Also Figure 11 shows how the normal branch Hubble parameter evolves on the brane with a phantom field. Also Figure 12 shows the behavior of $w_{eff}(z)$. There is a crossing of the phantom divide line in this setup. Therefore, with a minimally coupled phantom field on the normal DGP setup, it is possible to cross the phantom divide line by the effective equation of state parameter of the dark energy. Note that as we have shown previously, this crossing was impossible with a quintessence field on the brane.

To investigate the classical stability of the solutions with phantom field in the normal DGP setup and within the $w_\varphi - w'_\varphi$ phase-plane approach, we adopt the same strategy as has been done for quintessence field. For this minimally coupled phantom field, we have

$$w'_\varphi = -3(1 - w_\varphi^2) + \lambda\sqrt{-3(1 + w_\varphi)\Omega_\varphi}(1 - w_\varphi). \quad (37)$$

Now the $w_\varphi - w'_\varphi$ phase-plane is divided into the following two regions

$$\begin{cases} w_\varphi < -1, & w'_\varphi < 3w_\varphi(1 + w_\varphi) & c_a^2 > 0 & (\text{region I}) \\ w_\varphi < -1, & w'_\varphi > 3w_\varphi(1 + w_\varphi) & c_a^2 < 0 & (\text{region II}) \end{cases} \quad (38)$$

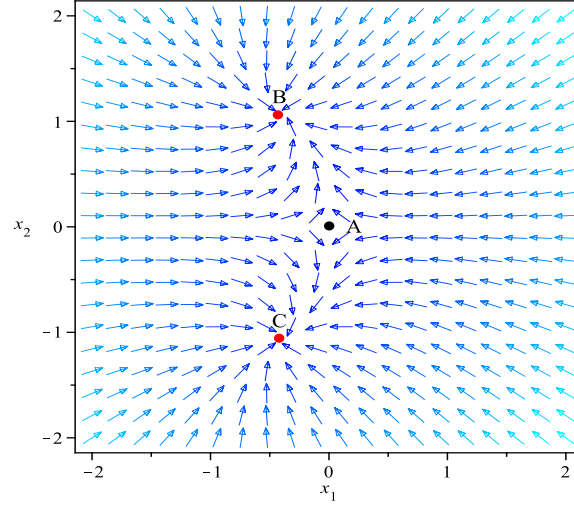


Figure 8. The phase plane for $\lambda = +1$ and $w_m = 0$. Point A is a saddle point, whereas points B and C are stable nodes .

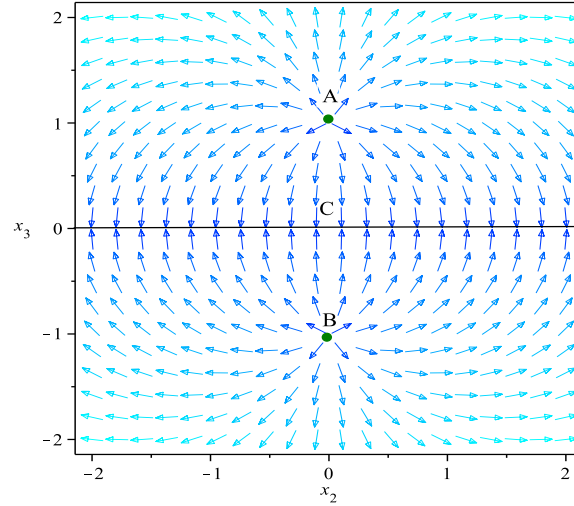


Figure 9. The $x_3 - x_2$ phase plane for $w_m = 0$. Points A and B are unstable nodes (which reflects the first row of table 2). There exists a critical line in this case.

The *Region I* is the subspace of the classical stability of the solutions with a minimally coupled phantom field on the brane.

3. Cosmological dynamics of a non-minimally coupled scalar field on the normal DGP setup

Now we consider a non-minimally coupled scalar field on the DGP brane. As we have stated previously, in a realistic gravitational or cosmological scenario with scalar fields,

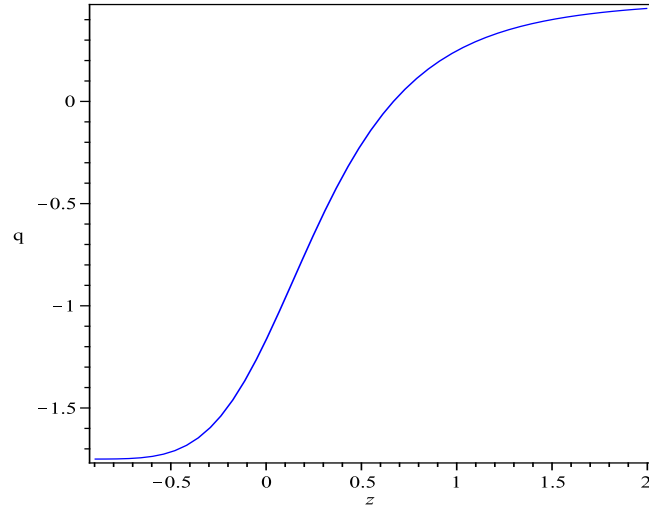


Figure 10. The deceleration parameter versus the redshift for $\Omega_m = 0.28$, $\Omega_\varphi = 0.8$, $w_m = 0$ and $w_\varphi = -1.5$. Transition to the accelerating phase occurs at $z = 0.67$.

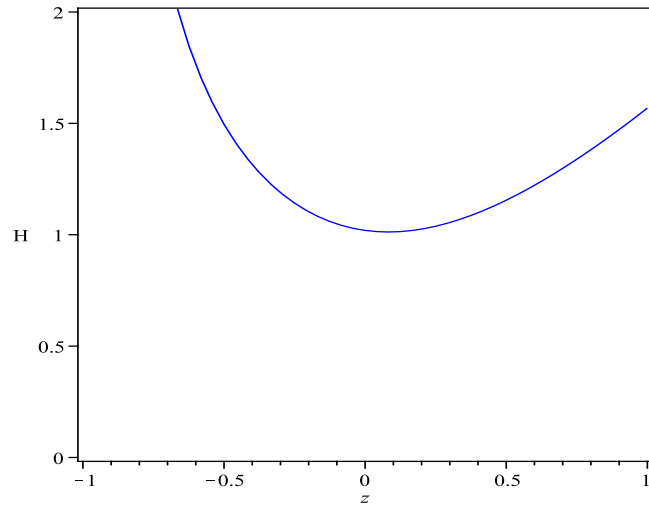


Figure 11. Evolution of the Hubble parameter with redshift on the brane with a phantom field. Here we considered $w_\varphi = -1.5$.

incorporation of the non-minimal coupling is inevitable. In fact, incorporation of a non-minimal coupling (NMC) between matter field and gravity is necessary from several compelling reasons. There are many theoretical evidences that suggest incorporation of an explicit non-minimal coupling of the scalar field and gravity in the action [20, 21]. A nonzero non-minimal coupling arises from quantum corrections and it is required also for renormalizability of the corresponding field theory. Amazingly, it has been proven that the phantom divide line crossing of the dark energy described by a single,

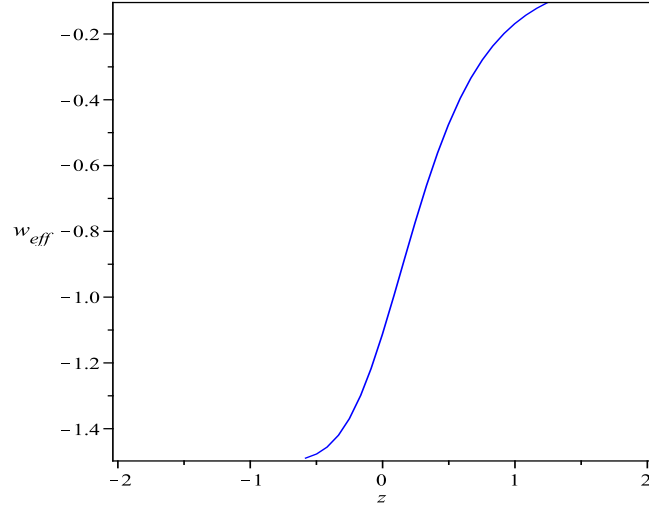


Figure 12. The effective equation of state parameter versus the redshift for $\Omega_m = 0.28$, $\Omega_\varphi = 0.8$, $w_m = 0$ and $w_\varphi = -1.5$. Transition to the phantom phase occurs at $z \sim 0.25$.

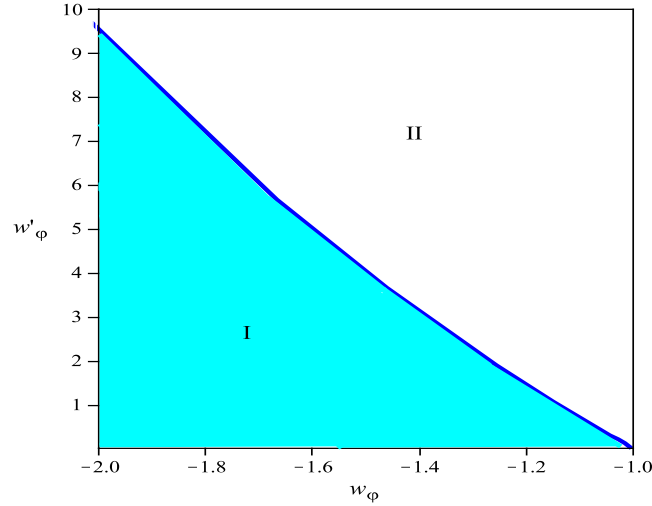


Figure 13. Bounds on w'_φ as a function of w_φ in $w_\varphi - w'_\varphi$ phase-plane for $\Omega_\varphi = 0.8$ and $\lambda = 0.1$. The subspace shown by *Region I* of the model parameter space is the classical stability subspace of the solutions.

minimally coupled scalar field with a general Lagrangian is even unstable with respect to the cosmological perturbations realized on the trajectories of the zero measure [22]. This fact has motivated a lot of attempts to realize crossing of the phantom divide line by equation of state parameter of a scalar field non-minimally coupled to gravity as dark energy candidate in more complicated frameworks [23]. In which follows, we study cosmological dynamics with a non-minimally coupled quintessence/phantom field on the normal DGP setup within a phase space approach analysis. We study possible

realization of the late-time acceleration and crossing of the phantom divide line in this setup. We also investigate the classical stability of the solutions in separate regions of the $w - w'$ phase-plane.

3.1. Non-minimally coupled quintessence field on the normal DGP branch

The equations governing on the cosmological dynamics on the normal DGP branch with a non-minimally coupled quintessence field are as follows

$$\dot{H} = -\frac{\rho + p}{2\mu^2} \left(1 + \frac{1}{2Hr_c}\right)^{-1}, \quad (39)$$

where $\rho = \rho_m + \rho_\varphi$ and $p = p_m + p_\varphi$, and the energy density and pressure of the scalar field are defined as [7]

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) + 6\xi H\varphi\dot{\varphi} + 3\xi H^2\varphi^2, \quad (40)$$

and

$$p_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) - 2\xi(\varphi\ddot{\varphi} + 2\varphi H\dot{\varphi} + \dot{\varphi}^2) - \xi\varphi^2(2\dot{H} + 3H^2), \quad (41)$$

respectively. As usual, ρ_m is the energy density of ordinary matter fields other than the scalar field φ on the brane. The other dynamical equation is the following Klein-Gordon equation

$$\ddot{\varphi} + 3H\dot{\varphi} + \xi R\varphi = -\frac{dV}{d\varphi}. \quad (42)$$

Now we define the dimensionless variables as

$$\begin{aligned} x_1 &= \frac{\dot{\varphi}}{\sqrt{6\mu}H}, & x_2 &= \frac{\sqrt{V}}{\sqrt{3\mu}H}, & x_3 &= \frac{\sqrt{\rho_m}}{\sqrt{3\mu}H}, \\ x_4 &= \frac{1}{\sqrt{2r_cH}}, & x_5 &= \frac{\sqrt{\xi}}{\mu}\varphi. \end{aligned} \quad (43)$$

By using these phase space variables, now the evolution equations (39) and (42) can be rewritten as

$$\dot{H} = -\frac{3(1 - 2\xi)x_1^2 + \sqrt{6\xi}x_1x_5 + \frac{3}{2}\gamma x_3^2 + 3\sqrt{\xi}x_5(\sqrt{6}x_1 + 4\sqrt{\xi}x_5 - \lambda x_2^2)}{1 + x_4^2 - (1 - 6\xi)x_5^2}H^2, \quad (44)$$

and

$$\begin{aligned} \ddot{\varphi} &= -3\mu H^2 \left[\frac{(-2\sqrt{\xi}x_5)(3(1 - 2\xi)x_1^2 + \sqrt{6\xi}x_1x_5 + \frac{3}{2}\gamma x_3^2)}{1 + x_4^2 - (1 - 6\xi)x_5^2} \right. \\ &\quad \left. + \frac{(1 + x_4^2 - x_5^2)(\sqrt{6}x_1 + 4\sqrt{\xi}x_5 - \lambda x_2^2)}{1 + x_4^2 - (1 - 6\xi)x_5^2} \right], \end{aligned} \quad (45)$$

respectively. Therefore, we obtain the following Friedmann constraint equation in the phase space of the model

$$x_1^2 + x_2^2 + x_3^2 + 2\sqrt{6\xi}x_1x_5 + x_5^2 - 2x_4^2 = 1. \quad (46)$$

To describe the dynamical system of the model, first we need to obtain the autonomous, phase space equations. We differentiate the phase space dimensionless variables with respect to $N = \ln a$ to find

$$\frac{dx_1}{dN} = -3x_1 + \frac{\sqrt{6}}{2}\lambda x_2^2 - 2\sqrt{6\xi}x_5 - (\sqrt{6\xi}x_5 + x_1)\Psi, \quad (47)$$

$$\frac{dx_2}{dN} = -\frac{\sqrt{6}}{2}\lambda x_1 x_2 - x_2 \Psi, \quad (48)$$

$$\frac{dx_4}{dN} = -\frac{1}{2}x_4 \Psi, \quad (49)$$

and

$$\frac{dx_5}{dN} = \sqrt{6\xi}x_1, \quad (50)$$

where by definition,

$$\Psi \equiv \left[-\frac{3(1-2\xi)x_1^2 + \sqrt{6\xi}x_1x_5 + \frac{3}{2}\gamma x_3^2 + 3\sqrt{\xi}x_5(\sqrt{6}x_1 + 4\sqrt{\xi}x_5 - \lambda x_2^2)}{1 + x_4^2 - (1-6\xi)x_5^2} \right].$$

The stability around the fixed points is related to the form of the eigenvalues in each critical point. The eigenvalues can be obtained by using the above autonomous equations, the results of which are as follows

- point (1a),(1b):

$$\alpha_{1,2} = \frac{3}{2}\gamma, \quad \alpha_{3,4} = -\frac{3}{2} + \frac{3}{4}\gamma \pm \frac{1}{4}\sqrt{36 - 36\gamma + 9\gamma^2 - 192\xi + 144\xi\gamma}$$

- point (2a),(2b):

$$\alpha_1 = -\frac{3}{2}\gamma + 2, \quad \alpha_2 = -1, \quad \alpha_{3,4} = 2$$

- curve C:

$$\alpha_1 = -\frac{3}{2}\gamma, \quad \alpha_2 = 0,$$

$$\begin{aligned} \alpha_{3,4} = \frac{1}{2} \frac{1}{(16\xi + 16x_2^2\xi - \lambda^2x_2^4 + 12\lambda^2x_2^4\xi)} & \left[-48\xi - 48x_2^2\xi + 3\lambda^2x_2^4 - 36\lambda^2x_2^4\xi \right. \\ & \pm \left(2304\xi^2 + 4608x_2^2\xi^2 - 288\lambda^2x_2^4\xi + 1920\lambda^2x_2^4\xi^2 + 2304x_2^4\xi^2 - 288x_2^6\xi\lambda^2 \right. \\ & + 4992x_2^6\xi^2\lambda^2 + 9\lambda^4x_2^8 - 72\lambda^4x_2^8\xi + 1872\lambda^4x_2^8\xi^2 - 24576x_2^2\xi^3 - 12288\xi^3 \\ & + 384\lambda^4x_2^6\xi - 3072\lambda^2x_2^2\xi^2 - 12288x_2^4\xi^3 - 12\lambda^6x_2^{10} - 9216x_2^6\xi^3\lambda^2 \\ & \left. \left. - 9216\lambda^2x_2^4\xi^3 + 144\lambda^6x_2^{10}\xi - 2304\lambda^4x_2^6\xi^2 \right)^{\frac{1}{2}} \right] \end{aligned}$$

Table 4. Location and dynamical character of the fixed points.

| name | x_{1c} | x_{2c} | x_{3c} | x_{5c} | stability | γ_φ | w_{eff} | $a(t)$ |
|-----------|----------|----------|----------|-------------------------------------|------------------|------------------|--------------|------------------------------------|
| (1a),(1b) | 0 | 0 | ± 1 | 0 | saddle point | undefined | $\gamma - 1$ | $a_0(t - t_0)^{\frac{2}{3\gamma}}$ |
| (2a),(2b) | 0 | 0 | 0 | ± 1 | saddle point | 4/3 | 1/3 | $a_0(t - t_0)^{\frac{1}{2}}$ |
| C | 0 | x_2 | 0 | $\frac{\lambda x_2^2}{4\sqrt{\xi}}$ | stable attractor | 0 | -1 | $e^{\Lambda(t-t_0)}$ |

We note that when there is a zero eigenvalue for a critical point, it is necessary to use the *center manifold theory* in order to study the stability of that point in phase space of the model. In our case there is a zero eigenvalue for a *critical line*, and therefore there is no need to do the center manifold analysis. In other words, since we have a critical line here, the non-vanishing eigenvalues are enough in this case to treat the stability of the critical points (see for instance [24]). Table 4 summarizes the results of the stability analysis in the phase space of this model. Also this table contains types of possible cosmological dynamics in this setup.

For critical points $\{(1a), (1b)\}$, there is no contribution of the scalar field and the universe is dominated by matter fields other than the quintessence scalar field. These two critical points behave like saddle points in the phase space. For $\gamma < 2/3$, one can obtain an accelerating phase of expansion, but this phase is not stable. It is possible to have scaling solutions for these cases. As an interesting case, if $\gamma = 2/3$, then we find $w_{eff} = -1/3$ which shows domination of the curvature energy. The critical points $\{(2a), (2b)\}$ also behave like saddle points in the phase space and in these cases we have no late-time acceleration. For these points, the universe is radiation dominated. The last line of table 4, stands for a critical line (C) and in this case, there is a cosmological constant dominated accelerating phase. In this case, the potential energy of the brane scalar field plays the role of a cosmological constant on the brane. So, with a non-minimally coupled quintessence scalar field on the DGP brane, it is possible to realize a stable, de Sitter late-time accelerating phase even in the normal branch of the model. Figure 14 shows the phase plane of the model with $\xi = 1/6$ (the conformal coupling). Point C is a stable attractor, whereas points A and B are saddle points.

Figure 15 shows the three-dimensional, $x_1 - x_2 - x_5$ phase space of the model for the last row of table 4. The stable, attractor points are located in a hyperbolic curve (curve C of table 4) in $x_2 - x_5$ perspective. In the three dimensional $x_1 - x_2 - x_5$ phase space, this is a hyperbolic hypersurface of stability points as shown in figure 15. All points of this hypersurface are stable attractors. We note that point C of figure 14 is corresponding to the mentioned hyperbolic on $x_1 - x_5$ plane.

Figure 16 shows the possibility of phantom divide crossing by the equation of state parameter of the model. The universe transits into the phantom phase from a quintessence phase in a redshift that is observationally viable ($z \approx 0.25$).

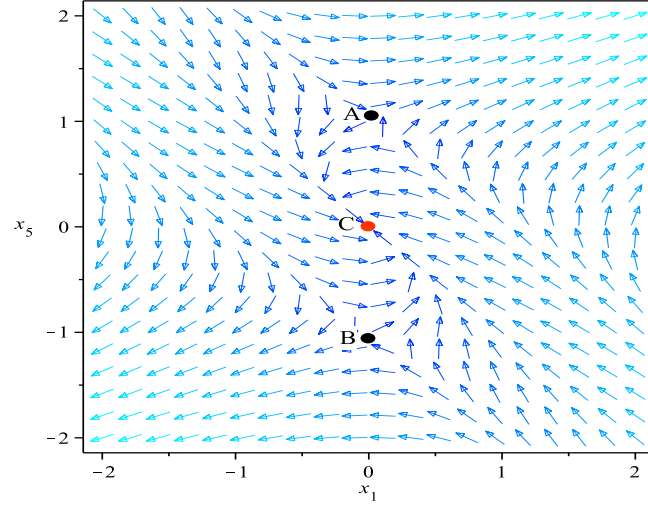


Figure 14. The phase plane for $\xi = 1/6$. Point C is a stable attractor, but points A and B are saddle points.

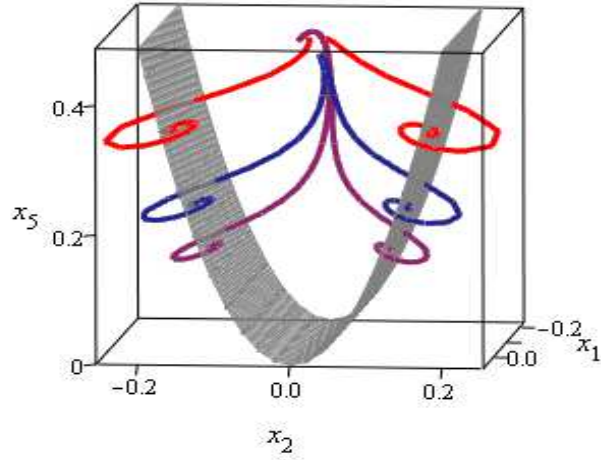


Figure 15. The 3-dimensional phase plane for $\xi = 1/6$. The hyperbolic surface contains the attractor, stable phases of the model.

From another perspective, the classical stability of the solutions in $w_\varphi - w'_\varphi$ phase-plane gives some interesting results. To do this end, we calculate w_φ versus w'_φ as

$$w'_\varphi = \left[4\sqrt{6\xi}(1-6\xi)(1+w_\varphi)bx_1x_5 - 3(1-6\xi)bw_\varphi(1+w_\varphi)x_5^2 \right. \\ \left. + \frac{((-6+28\xi)x_1^2 + 2\sqrt{6}\lambda(1-3\xi)x_1x_2^2 - 8\sqrt{6\xi}(1-4\xi)x_1x_5 + 2\sqrt{\xi}\lambda x_2^2x_5 - 8\xi x_5^2 + 2\sqrt{6\xi}\lambda^2x_1x_2^2x_5)}{x_1^2 + x_2^2 + x_5^2 + 2\sqrt{6\xi}x_1x_5} \right]$$

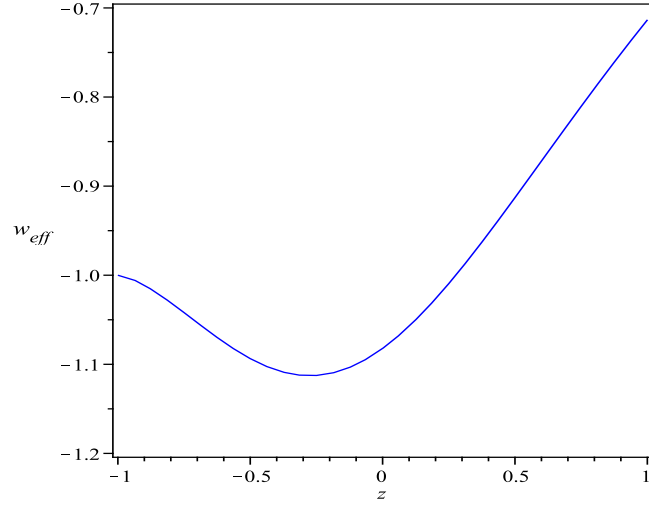


Figure 16. Crossing the phantom divide by the equation of state parameter for $a(t) = a_0 e^{\nu t}$ and $\varphi = \varphi_0 e^{-\alpha t}$, where $a_0, \varphi_0, \nu > 0$ and $\alpha > 0$ are constants. This crossing of the phantom divide occurs at $z \approx 0.25$

$$-\frac{3}{2}(x_1^2 + x_2^2 + x_5^2 + 2\sqrt{6\xi}x_1x_5)b^3(1-6\xi)(1+w_\varphi)^2x_4^2x_5^2 + 3w_\varphi(1+w_\varphi)\Big](1-b(1-6\xi)x_5^2)^{-1} \quad (51)$$

where by definition

$$b \equiv \left(1 + \frac{1}{2Hr_c}\right)^{-1} = (1 + x_4^2)^{-1}.$$

In this case, the $w_\varphi - w'_\varphi$ phase-plane is divided into the following four regions

$$\left\{ \begin{array}{ll} w_\varphi > -1, & w'_\varphi > 3w_\varphi(1+w_\varphi) \implies c_a^2 < 0 \quad (\text{region I}) \\ w_\varphi < -1, & w'_\varphi > 3w_\varphi(1+w_\varphi) \implies c_a^2 > 0 \quad (\text{region II}) \\ w_\varphi > -1, & w'_\varphi < 3w_\varphi(1+w_\varphi) \implies c_a^2 > 0 \quad (\text{region III}) \\ w_\varphi < -1, & w'_\varphi < 3w_\varphi(1+w_\varphi) \implies c_a^2 < 0 \quad (\text{region IV}) \end{array} \right. \quad (52)$$

As we have explained in previous sections, the stability of the solutions requires $c_a^2 > 0$. So, the stability regions of the solutions in this case are the regions *II* and *III*. The region *II* corresponds to an effective phantom phase while region *III* is a quintessence phase. Figure 17 shows these regions in $w_\varphi - w'_\varphi$ phase-plane.

3.2. Non-minimally coupled phantom field on the normal DGP setup

For completeness of our analysis, now we consider a non-minimally coupled phantom field on the normal DGP setup. The energy density and pressure of this non-minimally

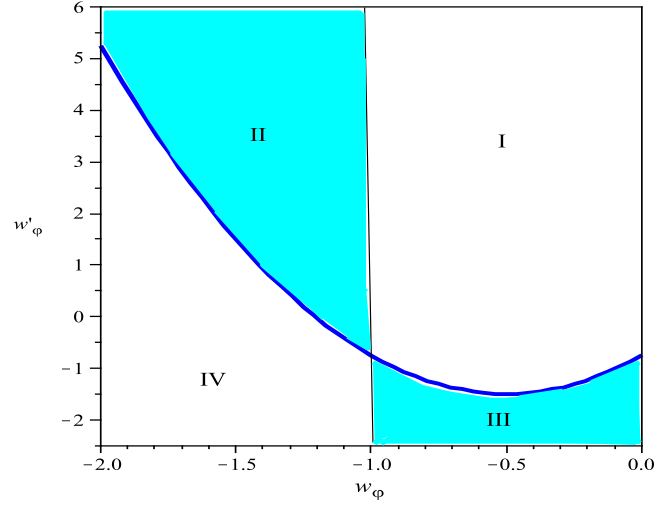


Figure 17. Bounds on w'_φ as a function of w_φ in $w_\varphi - w'_\varphi$ phase-plane for $\Omega_\varphi = 0.8$, $\xi = 1/6$ and $\lambda = 0.1$

coupled phantom field are given by

$$\rho_\varphi = -\frac{1}{2}\dot{\varphi}^2 + V(\varphi) + 6\xi H\varphi\dot{\varphi} + 3\xi H^2\varphi^2, \quad (53)$$

and

$$p_\varphi = -\frac{1}{2}\dot{\varphi}^2 - V(\varphi) - 2\xi(\varphi\ddot{\varphi} + 2\varphi H\dot{\varphi} + \dot{\varphi}^2) - \xi\varphi^2(2\dot{H} + 3H^2). \quad (54)$$

respectively. As previous cases, the equation governing on the the evolution of the Hubble parameter depends on the dimensionless variables and can be written as

$$\dot{H} = -\frac{3(-1 - 2\xi)x_1^2 + \sqrt{6\xi}x_1x_5 + \frac{3}{2}\gamma x_3^2 + 3\sqrt{\xi}x_5(\sqrt{6}x_1 - 4\sqrt{\xi}x_5 + \lambda x_2^2)}{1 + x_4^2 - (1 + 6\xi)x_5^2}H^2. \quad (55)$$

The Klien-Gordon equation for a non-minimally coupled phantom field is

$$\ddot{\varphi} + 3H\dot{\varphi} - \xi R\varphi = \frac{dV}{d\varphi}. \quad (56)$$

The Friedmann equation in terms of the phase space coordinates now can be written as the following constraint equation

$$-x_1^2 + x_2^2 + x_3^2 + 2\sqrt{6\xi}x_1x_5 + x_5^2 - 2x_4^2 = 1. \quad (57)$$

The parameter space of the model as a dynamical system is described by the following autonomous system

$$\frac{dx_1}{dN} = -3x_1 - \frac{\sqrt{6}}{2}\lambda x_2^2 + 2\sqrt{6\xi}x_5 + (\sqrt{6\xi}x_5 - x_1)\Delta, \quad (58)$$

$$\frac{dx_2}{dN} = -\frac{\sqrt{6}}{2}\lambda x_1x_2 - x_2\Delta, \quad (59)$$

$$\frac{dx_4}{dN} = -\frac{1}{2}x_4\Delta, \quad (60)$$

Table 5. Location and dynamical character of the fixed points.

| name | x_{1c} | x_{2c} | x_{3c} | x_{5c} | stability | γ_φ | w_{eff} | $a(t)$ |
|-----------|----------|----------|----------|-------------------------------------|--------------|------------------|--------------|------------------------------------|
| (1a),(1b) | 0 | 0 | ± 1 | 0 | saddle point | undefined | $\gamma - 1$ | $a_0(t - t_0)^{\frac{2}{3\gamma}}$ |
| (2a),(2b) | 0 | 0 | 0 | ± 1 | saddle point | 4/3 | 1/3 | $a_0(t - t_0)^{\frac{1}{2}}$ |
| C | 0 | x_2 | 0 | $\frac{\lambda x_2^2}{4\sqrt{\xi}}$ | saddle point | 0 | -1 | $e^{\Lambda(t-t_0)}$ |

$$\frac{dx_5}{dN} = \sqrt{6\xi}x_1. \quad (61)$$

where

$$\Delta \equiv \frac{\dot{H}}{H^2}.$$

As previous sections, in order to study the stability of the critical points, we should obtain their eigenvalues, which are written as follows:

- point (1a),(1b):

$$\alpha_{1,2} = \frac{3}{2}\gamma, \quad \alpha_{3,4} = -\frac{3}{2} + \frac{3}{4}\gamma \pm \frac{1}{4}\sqrt{36 - 36\gamma + 9\gamma^2 + 192\xi - 144\xi\gamma}$$

- point (2a),(2b):

$$\alpha_1 = -\frac{3}{2}\gamma + 2, \quad \alpha_2 = -1, \quad \alpha_{3,4} = 2$$

- curve C:

$$\alpha_1 = -\frac{3}{2}\gamma, \quad \alpha_2 = 0,$$

$$\alpha_{3,4} = \frac{1}{2} \frac{1}{(-16\xi - 16x_2^2\xi + \lambda^2x_2^4 + 12\lambda^2x_2^4\xi)} \left[48\xi + 48x_2^2\xi - 3\lambda^2x_2^4 - 36\lambda^2x_2^4\xi \right. \\ \left. \pm \left(2304\xi^2 + 4608x_2^2\xi^2 - 288\lambda^2x_2^4\xi - 1920\lambda^2x_2^4\xi^2 + 2304x_2^4\xi^2 - 288x_2^6\xi\lambda^2 \right. \right. \\ \left. \left. - 4992x_2^6\xi^2\lambda^2 + 9\lambda^4x_2^8 + 72\lambda^4x_2^8\xi + 1872\lambda^4x_2^8\xi^2 + 24576x_2^2\xi^3 + 12288\xi^3 \right. \right. \\ \left. \left. - 384\lambda^4x_2^6\xi + 3072\lambda^2x_2^2\xi^2 + 12288x_2^4\xi^3 + 12\lambda^6x_2^{10} - 9216x_2^6\xi^3\lambda^2 \right. \right. \\ \left. \left. - 9216\lambda^2x_2^4\xi^3 + 144\lambda^6x_2^{10}\xi - 2304\lambda^4x_2^6\xi^2 \right)^{\frac{1}{2}} \right]$$

Table 5 shows the results of the stability analysis in the phase space of a non-minimally coupled phantom field. In this case there are four critical points non of them result in a stable phase for the system. There is also a critical line, (C), which behaves as line of saddle points.

Figure 18 shows the $x_1 - x_5$ phase plane of the model with $\xi = 1/6$. In this case points A, B and C behave as saddle points.

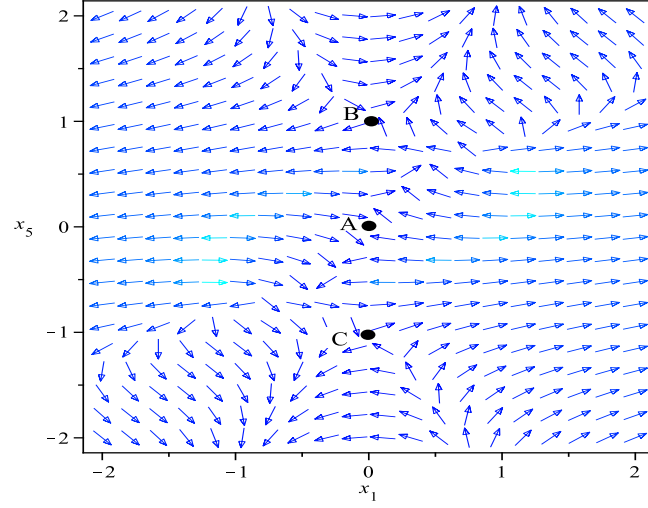


Figure 18. The $x_1 - x_5$ phase plane for a non-minimally coupled phantom field with $\xi = 1/6$. Points A , B and C are saddle points.

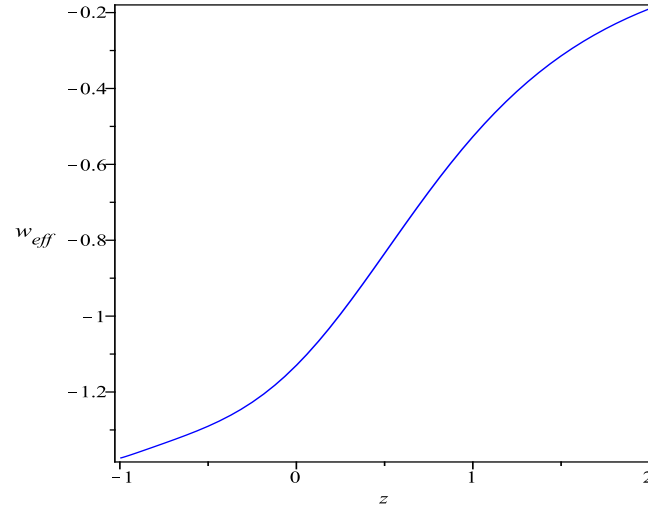


Figure 19. Crossing of the phantom line by the equation of state parameter for a phantom field on the normal DGP setup. We considered $a(t) = a_0 e^{\nu t}$ and $\varphi = \varphi_0 e^{-\alpha t}$, where a_0 , φ_0 , ν and α are positive constants. Crossing of the phantom divide occurs at $z \approx 0.24$

Now let us to study the possibility of phantom divide crossing by the equation of state parameter in this setup. As Figure 19 shows, in this case transition to the phantom phase occurs from quintessence phase at $z \approx 0.25$. Finally, investigation of classical stability of the solutions in $w_\varphi - w'_\varphi$ phase-plane gives interesting result. We calculate w_φ versus w'_φ for this model as

$$w'_\varphi = \left[4\sqrt{6\xi}(1+6\xi)(1+w_\varphi)bx_1x_5 - 3(1+6\xi)bw_\varphi(1+w_\varphi)x_5^2 \right] \quad (62)$$

$$\begin{aligned}
 & + \frac{\left((6 + 28\xi)x_1^2 + 2\sqrt{6}\lambda(1 + 3\xi)x_1x_2^2 - 8\sqrt{6\xi}(1 + 4\xi)x_1x_5 - 2\sqrt{\xi}\lambda x_2^2x_5 + 8\xi x_5^2 - 2\sqrt{6\xi}\lambda^2x_1x_2^2x_5 \right)}{-x_1^2 + x_2^2 + x_5^2 + 2\sqrt{6\xi}x_1x_5} \\
 & - \frac{3}{2}(-x_1^2 + x_2^2 + x_5^2 + 2\sqrt{6\xi}x_1x_5)b^3(1 + 6\xi)(1 + w_\varphi)^2x_4^2x_5^2 + 3w_\varphi(1 + w_\varphi) \Big] \left(1 - b(1 + 6\xi)x_5^2 \right)^{-1}
 \end{aligned}$$

The $w_\varphi - w'_\varphi$ phase-plane of the model is plotted in figure 20. This phase-plane can be divided into the following four regions

$$\left\{ \begin{array}{ll} w_\varphi > -1, & w'_\varphi > 3w_\varphi(1 + w_\varphi) \implies c_a^2 < 0 \quad (\text{region I}) \\ w_\varphi < -1, & w'_\varphi > 3w_\varphi(1 + w_\varphi) \implies c_a^2 > 0 \quad (\text{region II}) \\ w_\varphi > -1, & w'_\varphi < 3w_\varphi(1 + w_\varphi) \implies c_a^2 > 0 \quad (\text{region III}) \\ w_\varphi < -1, & w'_\varphi < 3w_\varphi(1 + w_\varphi) \implies c_a^2 < 0 \quad (\text{region IV}) \end{array} \right. \quad (63)$$

The stable regions of the phase-plane are those regions that the condition $c_a^2 > 0$ is fulfilled. In this case, these stable regions are Region II and III as shown in figure 20.

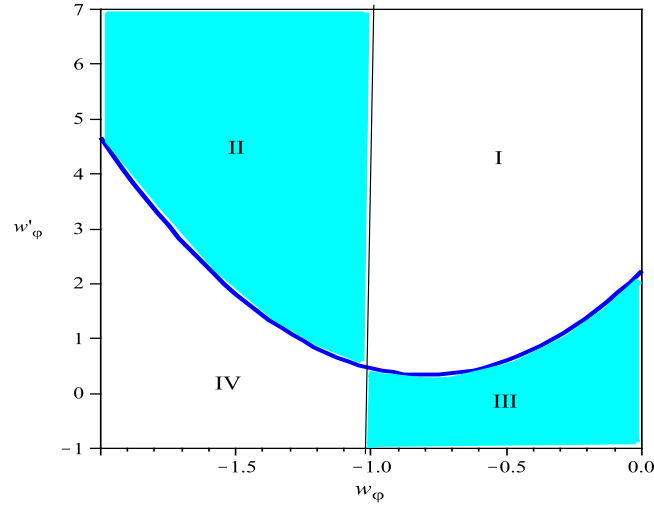


Figure 20. Bounds on w'_φ as a function of w_φ in $w_\varphi - w'_\varphi$ phase-plane for $\Omega_\varphi = 0.8$, $\xi = 1/6$ and $\lambda = 1$. The stability regions of the phase-plane are painted with cyan color.

4. Summary and Conclusion

The accelerated expansion of the universe supported by recent observational data could be associated with dark energy, whose theoretical nature and origin are still unknown for cosmologists. Cosmological constant or vacuum energy with an equation of state

parameter $\omega = -1$, is the most popular candidate for dark energy, but unfortunately it suffers from some serious problems such as huge fine-tuning and coincidence problems. Therefore, a number of models containing dynamical dark energy have been proposed as responsible mechanisms for late-time cosmic speed up. Some of these models are quintessence, k-essence, phantom scalar field, chaplygin gas models and so on. Another alternative approach to explain the late-time cosmic speed up is modification of the geometric sector of the Einstein field equations leading to modified gravity theories. In the spirit of modified gravity proposal, the Dvali-Gabadadze-Porrati (DGP) braneworld scenario explains the late-time accelerated expansion in its self-accelerating branch without need to introduce a dark energy component on the brane. However, some important features of dark energy such as possible crossing of the cosmological constant equation of state parameter are missing in the pure DGP model. In addition, the self-accelerating DGP solution suffers from ghost instability which makes the model unfavorable. Incorporation of a scalar field component on the DGP brane and treating the normal branch solutions brings a lots of new physics, some of which are studied in this paper. Previous studies in this field are restricted to either scalar fields on the self-accelerating branch or the simple case of minimally coupled scalar fields. In this paper we considered a scalar field component (both quintessence and phantom scalar fields), non-minimally coupled with induced gravity on the brane. We studied cosmological dynamics of the normal branch solutions on the brane within a dynamical system approach. We translated dynamical equations into an autonomous dynamical system in each case. Then we obtained the critical points of the model in phase space of each model. The issue of stability of these solutions are studied with details. Also possibility of having a stable attractor in de Sitter phase corresponding to current accelerated phase of universe expansion are studied in each case. We have also investigated the possibility to have a transition to the phantom phase of the equation of state parameter in each case. The classical stability of the solutions are treated also in a $w_\varphi - w'_\varphi$ phase-plane analysis in each step. We have shown in each step that there are several phases of accelerated expansion in each case, but only a limited critical points have stable, attractor solutions with de Sitter scale factor describing the current accelerated expansion on the brane. In summary, the main achievements of this study are as follows:

- While the pure, normal DGP solution has not the potential to explain late-time cosmic acceleration, with a minimally coupled quintessence field in the normal DGP setup there is a stable de Sitter phase realizing the late-time cosmic speed up. Nevertheless, as the pure DGP case, there is no possibility to cross the cosmological constant line by the effective equation of state parameter of the model. The classical stability domain of the model is restricted to those subspaces of the model parameter space that $w'_\varphi < 3w_\varphi(1 + w_\varphi)$ with $w_\varphi > -1$ where $w'_\varphi \equiv \frac{dw_\varphi}{dN}$ and $N \equiv \ln a(t)$.

- With a minimally coupled phantom field on the brane, it is possible to have an attractor, de Sitter solution realizing the late-time accelerated expansion in the nor-

mal DGP setup. Also, the effective equation of state parameter of the model crosses the phantom divide. Note that this crossing is impossible by the effective equation of state parameter of a minimally coupled quintessence field on the brane. Similar to the minimally quintessence field on the brane, the classical stability domain of the model is restricted to those subspaces of the model parameter space that $w'_\varphi < 3w_\varphi(1 + w_\varphi)$ with $w_\varphi < -1$.

- With a non-minimally coupled quintessence scalar field on the DGP brane, it is possible to realize a stable, de Sitter late-time accelerating phase in the normal branch of the model. It is possible also to cross the phantom divide by the effective equation of state parameter of the model. In this case, there are two different domains of stability in the $w_\varphi - w'_\varphi$ phase-plane of the model: a subspace with $w'_\varphi > 3w_\varphi(1 + w_\varphi)$ with $w_\varphi < -1$ corresponding to an *effective phantom phase* and the other subspace with $w'_\varphi < 3w_\varphi(1 + w_\varphi)$ with $w_\varphi > -1$ corresponding to a quintessence phase. This feature shows that it is possible to have an effective phantom picture with a non-minimally quintessence field on the normal DGP setup.

- For a non-minimally coupled phantom scalar field on the DGP brane, it is possible to realize a de Sitter late-time accelerating phase in the normal branch of the model. It is possible also to cross the phantom divide line by the effective equation of state parameter of the model in this case. Also, as for the case of non-minimally coupled quintessence field on the brane, there are two different domains of stability in the $w_\varphi - w'_\varphi$ phase-plane of the model: a subspace with $w'_\varphi > 3w_\varphi(1 + w_\varphi)$ with $w_\varphi < -1$ corresponding to the phantom phase and the other subspace with $w'_\varphi < 3w_\varphi(1 + w_\varphi)$ with $w_\varphi > -1$ corresponding to an *effective quintessence phase* on the brane. Note that it is possible to have an effective quintessence picture with a non-minimally coupled phantom field on the normal DGP setup.

Finally we stress that the observational status of the present DGP-inspired models can be treated in the same line as has been reported in Ref. [25].

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